CONCERNING POINTWISE LIMITS OF SEQUENCES OF FUNCTIONS

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Simple graphs which are pointwise limits of sequences of continuous simple graphs and simple graphs which are pointwise limits of sequences of quasi-continuous simple graphs each continuous on the right are characterized in this paper. These characterizations depend upon Theorem I which deals with certain collections of point sets. In order to state Theorem I, it will be convenient to introduce the words "acceptable" and "determines."

Definition. If K is a collection of point sets, the subcollection K' of K is said to be an acceptable subcollection of K if and only if it is true that if u is an element of K' and v is an element of K and v is not a subset of u then v is in K'.

The word determines is not defined but is assumed to have the following three properties:

Property (1). If H is a collection of point sets which determines h, then h is a point set which is a subset of each member of H.

Property (2). If the collection K of point sets determines a point set k, then each subcollection of K determines a set which is not a proper subset of k.

Property (3). No collection of point sets determines two point sets.

Theorem I. If M is a point set, then there is only one collection H of point sets which has properties (a), (b), (c), (d), and (e) listed below. Moreover, if the statement x <u>precedes</u> y means y is a proper subset of x, then H is well ordered with respect to this meaning of the word precedes.

Property (a). M is in H.

Property (b). Each member of H is a subset of M.

Property (c). If an acceptable subcollection H' of H determines D, and no acceptable proper subcollection of H' determines a member of itself then D is in H.

Property (d). If D is a member of H distinct from M, then there is only one acceptable subcollection H' of H which determines D, does not contain D, and has no acceptable proper subcollection which determines a member of itself.

Property (e). If there is some set w which is determined by the acceptable subcollection H' of H and D is in H, then D is in H' or D is a subset of w.

The following property is also introduced:

Property (c'). If an acceptable <u>proper</u> subcollection H' of H determines D, and no acceptable proper subcollection of H' determines a member of itself then D is in H.

Notice that in the following proof of Theorem I, the axiom of choice is not called upon. A collection with properties (a), (b), (c), (d), and (e) is said to have property (f). A collection with properties (a), (b), (c'), (d), and (e) is said to have property (f').

Proof of Theorem I. The collection whose only member is M has property (f'). We shall now assume that G is a collection which has property (f'), and introduce the notation that if J is an element of G distinct from M, then G_J denotes the only acceptable subcollection of G which determines J, does not contain J, and has no acceptable proper subcollection which determines a member of itself. It will now be shown that G has the following properties labeled (A), (B), (C), (D), (E), (F), and (G).

(A) If J and K belong to G, $J \neq M$, $K \neq M$, and G_K is not a subcollection of G_J, then K is a proper subset of J.

Statement (A) follows from (e) and (1).

(B) If P and Q belong to G and are distinct from M and P is a proper subset of Q, then G_Q is a proper subcollection of G_P.

Proof. If P and Q are members of G distinct from M and P is a proper subset of Q then by (1), P is a proper subset of each member of G_Q and by (e), G_Q is a subcollection of G_P. But by (3), $G_Q \neq G_P$ therefore G_Q is a proper subcollection of G_P.

(C) G is monotonic.

Proof. If J and K are elements of G, and J is not a subset of K then, by (e), J is in G_K and, by (1), K is a subset of J.

(D) If the statement x precedes y means y is a proper subset of x, then G is well ordered with respect to this meaning of the word precedes.

Proof. Let the statement x precedes y mean y is a proper subset of x. If J precedes K and K precedes L, then J precedes L. If J precedes L, then L does not precede J. It follows from (C) that if J and K are two members of G, then J precedes K or K precedes J. It will now be established that every subcollection of G has a first term. Suppose that there is some subcollection G' of G which has no first term. Denote by G" the collection of all members of G which precede each member of G'. It follows from (b) that M is in G". If w is in G', then there is a member y of G' which precedes w. From (e) we see that y is in G_w and, since y is in G', y is a subset of each member of G" and therefore, by the definition of acceptable, we see that each member of G" is in G_w. G" is acceptable and no acceptable subcollection of G_w determines a member of itself. Because of (2), G" determines a set V. The set V is not in G" because G" is an acceptable subcollection of G_w. Because of (c) we see that V is in G. V is not in G' because, as we have seen, each member of G' is determined by a collection which is an acceptable subcollection of G and which contains G_V as a proper subcollection. Since V is not in G", there is a member U of G' which is not a subset of V and therefore, by (C), V is a proper subset of U. From (B) we see that G_U is a proper subcollection of G_V. But G' is G_V and G_V is a proper subcollection of G_U. Therefore our assumption that G' does not have a first term is false.

(E) If x is a member of G distinct from M and G'_x is the collection of all the members of G which precede x, then G'_x is G_x.

Proof. Suppose x is a member of G distinct from M and G'_x is the collection of all the members of G which precede x. Suppose now that G_x contains a member w which is not in G'_x. The set w is not x because x is not in G_x and w is in G_x. The set w is a proper subset of x and, since G_x is acceptable, G_x must contain x. This is a contradiction to the assumption that G_x is not a subcollection of G'_x. Therefore G_x is a subcollection of G'_x. Now suppose that G'_x contains a member w of G which is not in G_x, then, because of (e), W is a subset of each member of G_x. Therefore, w is a subset of x but, since no member of G'_x is a subset of x, we see that our assumption is false and G'_x is a subcollection of G_x.

(F) If the collection H of sets has property (f') then G is an acceptable subcollection of H or H is an acceptable subcollection of G.

Proof. Suppose that there is a collection of sets H which has property (f') such that H is not an acceptable subcollection of G and G is not an acceptable subcollection of H. Suppose now that H contains a set which is not in G. Denote by x the first set of H which is not in G. H_x contains M and each member of H_x is a member of G. Let G' denote the subcollection of G to which y belongs if and only if there is a member z of H_x such that y is not a proper subset of z. If each member of G' is in H, then G' is an acceptable subcollection of H and, since G is not an acceptable subcollection of H, G' $\neq$ G and, by (c), x is in G. Therefore there is a member of G' which is not in H. Denote by w the first member of G' which is not in H. The collection G_w is an acceptable subcollection of H_x and, since no acceptable subcollection of H_x determines a member of itself, it follows from (c) that w is in H. This constitutes a contradiction, therefore G has property (F).

(G) If L is an acceptable proper subcollection of G, then L has property (f').

Proof. If L is an acceptable proper subcollection of G, then L has properties (a) and (b) and since each acceptable proper subcollection of L is an acceptable proper subcollection of G, it follows that L has properties (c') and (e). If D is in L, then D is in G and G_D is an acceptable proper subcollection of both L and G, therefore L has property (d).

Let K denote the collection to which the set x belongs if and only if x is a member of a collection which has property (f'). We see at once that K has properties (a) and (b).

We shall now show that K has property (c') by showing that if K' is an acceptable proper subcollection of K then K' determines a member of K.

If K' is an acceptable proper subcollection of K, then there is some member y of K which is not in K'. The set y belongs to some collection R with property (f'). Let x denote the first member of R which is not in K'. Since x is in R, x is in K. From (E) it follows that R_x is a subcollection of K'. From (G) it follows that R_x has property (f'). Suppose that there is a set w which is in K' but not in R_x. There is a collection T with property (f') which contains w. From (G), we see that T_w must have property (f'). From (F), it follows that T_w is an acceptable proper subcollection of R_x or R_x is an acceptable proper subcollection of T_w. Since T_w determines w, w is not in R_x, and R_x has property (c'), it cannot be true that T_w is an acceptable proper subcollection of R_x. Therefore R_x is an acceptable proper subcollection of T_w. From (A) we see that w is a proper subset of x. Since x is in K but not in K', x is a subset of each member of K' by the meaning of the word acceptable. Since w is in K' and w is a proper subset of x, w is a proper subset of itself. The assumption that K' is not a subcollection of R_x has therefore led to a contradiction. Since K' is a subcollection of R_x and R_x is a subcollection of K', K' is R_x. Therefore K' determines x. Therefore K has property (c').

It will now be established that K has property (d).

Let D denote a member of K which is distinct from M. Denote by K' the collection of all the members of K which are not subsets of D. D is in a collection R which has property (f'). R_D is a subcollection of K'. Suppose that there is some element w of K' which is not in R_D. The set w is in a collection T with property (f'). It follows from (F) that R_D is an acceptable proper subcollection of T_w or T_w is an acceptable proper subcollection of R_D. The collection T_w is not an acceptable proper subcollection of R_D because w is not in R_D. The collection R_D is not an acceptable proper subcollection of T_w because D is not in T_w. This constitutes a contradiction. Therefore, each member of K' is in R_D. K' is the only acceptable subcollection of K which determines D, does not contain D, and has no acceptable proper subcollection which determines a member of itself. Therefore K has property (d).

K' is the only acceptable subcollection of K which determines D, and we have seen that if x is in K then x is in K' or x is a subset of D. Therefore K has property (e).

The supposition that K determines a set x which does not belong to K leads to a contradiction because the set which consists only of the elements of K along with the element x has property (f'). Therefore K has property (c).

Suppose that there is a set J which has property (f) and is distinct from K. We see from (F) that J is an acceptable proper subcollection of K or K is an acceptable proper subcollection of J. Suppose that J is an acceptable proper subcollection of K. Denote by x the first member of K which is not in J. We see from (E) that J is the only acceptable subcollection of K which determines x, does not contain x, and has no acceptable proper subcollection which determines a member of itself. Therefore J does not have property (c'). Therefore K is the only collection which has property (f). This completes the proof of Theorem I.

Notation. If h is a number, let $|h$ denote the vertical line which contains the point (h, h) and let $\underline{h}$ denote the horizontal line which contains the point (h,h).

The following definitions facilitate the discussion of Theorem II:

Definition. The statement that the simple graph f with x-projection [a,b] has property c_1 means that f is the pointwise limit of a sequence of continuous simple graphs each with x-projection [a,b].

Definition. The statement that the simple graph f with x-projection [a,b] has property c_2 over the segment s means that if c is a number, there exist sequences $T_1(c)$,

$T_2(c)$ .... and $B_1(c)$, $B_2(c)$, ... which satisfy the following three conditions:

(1) If n is a positive integer, $T_n(c)$ and $B_n(c)$ are finite collections of intervals.

(2) If n is a positive integer, $[T_n(c)]^*$ and $[B_n(c)]^*$ are mutually exclusive.

(3) If u is a point of [a,b] which is in s and $f(u) > c$ and v is a point of [a,b] which is in s and $f(v) < c$, then there exists a positive integer K such that if L is a positive integer greater than K then u is in $[T_L(c)]^*$ and v is in $[B_L(c)]^*$.

Definition. The statement that the simple graph f with x-projection [a,b] has property c_2 at [x,f(x)] means that there is a segment s containing x such that f has property c_2 over s.

Definition. The statement that the simple graph f with x-projection [a,b] has property c_3 over the segment s means that if c and d are numbers and $c > d$, then there exist sequences $T_1(c,d)$, $T_2(c,d)$, ... and $B_1(c,d)$, $B_2(c,d)$, ... which satisfy the following conditions:

(1) If n is a positive integer, $T_n(c,d)$ and $B_b(c,d)$ are finite collections of intervals.

(2) If n is a positive integer, $T_n(c,d)]^*$ and $[B_n(c,d)]^*$ are mutually exclusive.

(3) If u is a point of [a,b] which is in s and $f(u) > c$ and v is a point of [a,b] which is in s and

$f(v) < d$, then there exists a positive integer K such that if L is a positive integer greater than K, u is in $[T_L(c,d)]^*$ and v is in $[B_L(c,d)]^*$.

Definition. The statement that the simple graph with x-projection $[a,b]$ has property c_3 at the point $[x, f(x)]$ means that there exists a segment s containing x such that f has property c_3 over s.

Definition. The statement that the simple graph f with x-projection $[a,b]$ has property c_4 at the point p of f means that there exist two vertical lines h and k with p between them such that if α and β are two horizontal lines and T is a subset of $[a,b]$ having the property that each point of f with abscissa in T is between h and k and above α and β, and B is a subset of $[a,b]$ having the property that each point of f with abscissa in B is between h and k and below α and β, then there is a point of B which is not a limit point of T or there is a point of T which is not a limit point of B.

Theorem II. If f is a simple graph with x-projection $[a,b]$, then each two of the following four statements are equivalent:

(1) f has property c_1.

(2) f has property c_2 at each of its points.

(3) f has property c_3 at each of its points.

(4) f has property c_4 at each of its points.

The following definitions facilitate the discussion of theorem III:

Definition. The statement that the simple graph f with x-projection [a,b] has property r_1 means that f is the pointwise limit of a sequence of continuous on the right quasi-continuous simple graphs.

Definition. The statement that the simple graph f with x-projection [a,b] has property r_2 over the segment s means that if c is a number, there exist sequences $T_1(c)$, $T_2(c)$, ... and $B_1(c)$, $B_2(C)$, ... which satisfy the following three properties:

(1) If n is a positive integer, $T_n(c)$ and $B_n(c)$ are finite collections of closed on the left sects.

(2) If n is a positive integer, $[T_n(c)]^*$ and $[B_n(c)]^*$ are mutually exclusive.

(3) If u is a point of [a,b] which is in s and $f(u) > c$ and v is a point of [a,b] which is in s and $f(v) < c$, then there is a positive integer K such that if L is a positive integer greater than K then u is in $[T_L(c)]^*$ and v is in $[B_L(c)]^*$.

Definition. The statement that the simple graph f with x-projection [a,b] has property r_2 at the point [x,f(x)] means that there is a segment s containing x such that f has property r_2 over s.

Definition. The statement that the simple graph f with x-projection [a,b] has property r_3 over the segment s means that if c and d are numbers and $c > d$ then there exist sequences $T_1(c,d)$, $T_2(c,d)$, ... and $B_1(c,d)$, $B_2(c,d)$, ... which satisfy the following properties:

(1) If n is a positive integer, $T_n(c,d)$ and $B_n(c,d)$ are finite collections of closed on the left sects.

(2) If n is a positive integer, $[T_n(c,d)]^*$ and $[B_n(c,d)]^*$ are mutually exclusive.

(3) If u is in $[a,b]$ and $f(u) > c$ and v is in $[a,b]$ and $f(v) < d$, then there is a positive integer K such that if L is a positive integer greater than K, then u is in $[T_L(c,d)]^*$ and v is in $[B_L(c,d)]^*$.

Definition. The statement that the simple graph f with x-projection $[a,b]$ has property r_3 at the point $[x,f(x)]$ means that there exists a segment s containing x such that f has property r_3 over s.

Definition. The statement that the simple graph f with x-projection $[a,b]$ has property r_4 at the point p of f means that there exist two vertical lines h and k with p between them such that if α and β are two horizontal lines and T is a subset of $[a,b]$ having the property that each point of f with abscissa in T is between h and k and above α and β and B is a subset of $[a,b]$ having the property that each point of f with abscissa in B is between h and k and below α and β, then there is a point of B which is not a limit point of T from the right or there is a point of T which is not a limit point of B from the right.

Theorem III. If f is a simple graph with x-projection [a,b], then each two of the following four statements are equivalent:

(1) f has property r_1.

(2) f has property r_2 at each of its points.

(3) f has property r_3 at each of its points.

(4) f has property r_4 at each of its points.

The following lemma will be used to show that statements (1) and (2) of theorem II are equivalent.

Lemma. If f is a simple graph with x-projection [a,b] and f has property c_2 at each of its points, then there is a segment s which contains [a,b] such that f has property c_2 over s.

Proof of the lemma. Suppose f is a simple graph with x-projection [a,b] and f has property c_2 at each of its points. If x is a number in [a,b], there exists a segment s_x containing x such that f has property c_2 over s_x.

There exists a finite sequence of segments $S = s_1, s_2, \ldots s_k$ which has the following four properties:

(1) If s_n is in S, then f has property c_2 over s_n.

(2) S covers [a,b].

(3) No proper subsequence of S covers [a,b].

(4) If n is a positive integer less than k, then the left end of s_n lies to the left of the left end of s_{n+1}.

Suppose c is a number. Let $T_1(c)$, $T_2(c)$, ... and $B_1(c)$, $B_2(c)$, ... satisfy the definition of property c_2 over s_1. Let $T_1'(c)$, $T_2'(c)$ ... and $B_1'(c)$, $B_2'(c)$, ... satisfy the definition of property c_2 over s_2. The set $s_1 \cdot s_2$ is a segment. Either $f(x) = c$ for each x is $s_1 \cdot s_2$ or there is a number x in $s_1 \cdot s_2$ such that $f(x) \neq c$.

Suppose $f(x) = c$ for each number x in $s_1 \cdot s_2$. Denote by I and J two mutually exclusive intervals which are subsets of $s_1 \cdot s_2$. Denote by $U_n(c)$ the set to which x belongs if and only if x is in I or x is in $[\overline{s_1 - s_1 \cdot s_2}]$ and $[T_n(c)]^*$ or x is in $[\overline{s_2 - s_1 \cdot s_2}]$ and $[T_n'(c)]^*$ Denote by $V_n(c)$ the set to which x belongs if and only if x is in J or x is in $[\overline{s_1 - s_1 \cdot s_2}]$ and $[B_n(c)]^*$ or x is in $[s_2 - s_1 \cdot s_2]$ and $[B_n'(c)]^*$. For each positive integer n, U_n is the sum of a finite collection of intervals t_n and V_n is the sum of a finite collection of intervals b_n; therefore f has property c_2 over $s_1 + s_2$.

Now suppose that there is a number x in $s_1 \cdot s_2$ such that $f(x) \neq c$. Suppose for convenience that $f(x) > c$. Let M denote some interval which does not intersect $[\overline{s_1 + s_2 + \ldots + s_k}]$. There is a positive integer n such that if m is a positive integer greater than n then x is in $[T_m(c)]^* \cdot [T'_m(c)]^*$. Therefore there is an interval L_m which contains x but contains no point of $[B_m(c)]^* + [B'_m(c)]^*$.

For each positive integer n, let U_n denote the number set to which t belongs if and only if t satisfies one of the following three conditions:

(1) The number t is in an interval I of $T_{m+n}(c)$ where I contains a number less than x and $t \leq x$.

(2) The number t is in an interval J of $T'_{m+n}(c)$ where J contains a number greater than x and $t \geq x$.

(3) The number t is in M.

For each positive integer n, let V_n denote the number set to which t belongs if and only if t satisfies one of the following three conditions:

(1) The number t is in an interval I of $B_{m+n}(c)$ where I contains a number less than x and $t \leq x$.

(2) The number t is in an interval I of $B'_{m+n}(c)$ where I contains a number greater than x and $t \geq x$.

(3) The number t is in M.

Each of U_n and V_n is the sum of a finite collection of intervals. f has property c_2 over s_1+s_2. Since S is finite, f has property c_2 over $s_1+s_2+ \ldots +s_k$. This completes the proof of the lemma.

Suppose f is a simple graph with x-projection [a,b]. It will now be established that if f has property c_1, then f has property c_2 at each of its points. Suppose f has property c_1. If c is a number such that each point of f is above or on <u>c</u> then there exist sequences $T_1(c)$, $T_2(c)$ ... and $B_1(c)$, $B_2(c)$, ... which have the three properties of the sequences in the definition of property c_2. It is also obvious that sequences of this type exist if c is a number such that each point of f is below or on c. Suppose c is a number such that there is a point [u, f(u)] above c and a point [v, f(v)] below c. There is a positive number ϵ such that [u, f(u)] is above $c+\epsilon$ and [v, f(v)] is below $c-\epsilon$. There exists a positive integer K such that for each positive integer $K' \geq K$, $f_{K'}(u) > c + \epsilon/2$ and $f_{K'}(v) < c - \epsilon/2$. Denote by S_n the collection to which the segment s belongs if and only if s satisfies one of the following three conditions:

(1) If x is in s, $f_{K+n}(x) > c + \epsilon/2^{n+1}$.

(2) If x is in s, $f_{K+n}(x) < c - \epsilon/2^{n+1}$.

(3) If x is in s, $c - \epsilon/2^n < f_{K+n}(x) < c + \epsilon/2^n$.

Since f is continuous, S_n covers [a,b]. Some finite subcollection S'_n of S_n covers [a,b]. For each positive integer n, let $T_n(c)$ denote the collection of intervals to which the interval I belongs if and only if there is some segment s in S'_n such that s contains the abscissa of a point of f_{K+n} above $c + \epsilon/2^n$ and $\bar{s} = I$. Let $B_n(c)$ denote the collection of intervals to which the interval I belongs if and only if there is some segment s in S'_n such that s contains the abscissa of a point of f_{K+n} below $c - \epsilon/2^n$ and $\bar{s} = I$. Notice that since $T_1(c)$, $T_2(c)$, ... and $B_1(c)$, $B_2(c)$, ... satisfy the three conditions imposed on the sequences in the definition of property c_2, f has property c_2 at each of its points.

It will now be established that if f has property c_2 at each of its points then f has property c_1. If f has property c_2 at each of its points then, by the lemma, there exists a segment S containing [a,b] such that f has property c_2 over S. For each number c, let $T_1(c)$, $T_2(c)$, ... and $B_1(c)$, $B_2(c)$, ... denote two sequences of intervals which satisfy the three conditions imposed upon the sequences in the definition of property c_2 over S and also satisfy the following six conditions for each positive integer n:

(1) If D is a nonnegative number, $T_n(D)$ contains [b+1, b+2].

(2) If D is a positive number, $B_n(D)$ contains [b+3, b+5].

(3) $T_n(0)$ contains [b+3, b+4].

(4) $B_n(0)$ contains [b+5, b+6].

(5) If D is a negative number, $T_n(D)$ contains [b+5, b+6].

(6) If D is a nonpositive number, $B_n(D)$ contains [b+7, b+8].

Suppose c is a positive rational number, c = r/s in lowest terms, and n is a positive integer. Denote by E(c) the collection to which the number D belongs if and only if D is a nonnegative rational number, D = u/v in lowest terms, $v \leq s$, and $D \leq c$.

Let $U_n(c)$ denote the number set to which the number x belongs if and only if, for each D in E(c), x is in

$[T_n(D)]^*$. Let $u_n(c)$ denote the number set to which the number x belongs if and only if there is a number D in $E_n(c)$ such that x is in $[T_n(0)]^* \cdot [B_n(D)]^*$. Let $\ell_n(c)$ denote the number set to which the number x belongs if and only if there is a number D in E(c) such that x is in $[B_n(0)]^* \cdot [T_n(-D)]^*$. Let $L_n(c)$ denote the number set to which the number x belongs if and only if, for each D in E(c), x is in $[B_n(-D)]^*$.

If c is a positive rational number and n is a positive integer, then there exists a finite collection $S_n(c)$ with the following properties:

(1) Each member of $S_n(c)$ is an interval or a degenerate point set.

(2) No two members of $S_n(c)$ have a point in common.

(3) $[S_n(c)]^* = [U_n(c) + u_n(c) + \ell_n(c) + L_n(c)]^*$.

Denote by M_n the number set to which the number x belongs if and only if x is the midpoint of a segment (d,e) which satisfies the following three conditions:

(1) The number d is a or d is an end point of a member of $T_n(0) + B_n(0)$.

(2) The number e is b or e is the endpoint of a member of $T_n(0) + B_n(0)$.

(3) There is no point of $[T_n(0) + B_n(0)]^*$ in (d,e).

Denote by $Q_n(c)$ the number set to which the number x belongs if and only if either x=a, x=b, x is the only member in a degenerate set of $S_n(c)$, x is an endpoint of an interval of $S_n(c)$, or x is in M_n.

Denote by $P_n(c)$ the point set to which the point (x,y) belongs if and only if either x is in $[Q_n(c)]\cdot[U_n(c)]^*$ and $y = c$, x is in $[Q_n(c)]\cdot[L_n(c)]^*$ and $y = -c$, or x is in $Q_n(c)$ but not in $[U_n(c) + L_n(c)]^*$ and $y = 0$.

Let f_c^n denote the contraction to $[a,b]$ of the polygon graph connecting the points of $P_n(c)$.

Let R denote a sequence of rational numbers r_1, r_2, ... which contains each rational number only once.

Let G denote the sequence g_1, g_2, ... of continuous simple graphs each with x projections $[a,b]$ and each satisfying the following three properties:

(1) If n is a positive integer and x is a member of M_n which is in $[a,b]$ then $g_n(x) = 0$.

(2) If n is a positive integer, x is a number which is in $[a,b]$ but not in M_n, e is the largest number of $M_n+\{a\}$ less than or equal to x, d is the smallest number of $M_n+\{b\}$ greater than or equal to x, and there is a point of $[T_n(0)]^*$ in (e,d) then $g_n(x)$ is the least upper bound of $[f^n_{r_1}(x), f^n_{r_2}(x), \ldots f^n_{r_n}(x)]$.

(3) If n is a positive integer, x is a number which is in $[a,b]$ but not in M_n, e is the largest number of $M_n+\{a\}$ less than or equal to x, d is the smallest number of $M_n+\{b\}$ greater than or equal to x, and there is a point of $[B_n(0)]^*$ in (e,d) then $g_n(x)$ is the greatest lower bound of $[f^n_{r_1}(x), f^n_{r_2}(x), \ldots f^n_{r_n}(x)]$.

Notice that each member of G is continuous. It will now be established that f is the pointwise limit of G.

Suppose ϵ is a positive number and x is a number in [a,b] such that $f(x) > 0$. There is a rational number r_n in R such that $r_n < f(x)$ and $f(x) - r_n < \epsilon$. Since $E(r_n)$ is finite and f has property c_2 over S, there exists a positive integer J such that if w is in $E(r_n)$ and K is a positive integer greater than J, then x is in $[T_K(w)]^*$. Notice that if L is a positive integer which is greater than J + n, then x is in $U_L(r_n)$, $f^L_{r_n}(x) = r_n$, and $g_L(x) \geq r_n$.

There is a rational number $t = r/s$ in lowest terms such that $t > f(x)$ and $t - f(x) < \epsilon$. Denote by Z the set of rational numbers to which the rational number $x = u/v$ in lowest terms belongs if and only if $v \leq s$, $u/v \geq t$, and $(u-1)/v < t$. If z is in Z, there exists a positive integer N such that if N' is a positive integer greater than N, x is in $B_{N'}(z)$. Since Z is finite, there exists a positive integer M such that if M' is a positive integer greater than M and w is in Z then x is in $B_M'(w)$. If P is a positive integer greater than K+n+M, then $|g_P(x) - f(x)| < \epsilon$. A similar argument can be used to establish this fact in case $f(x) < 0$ or $f(x) = 0$. Therefore if f has property c_2 at each of its points then f has property c_1.

It will now be demonstrated that if p is a point of f, f has property c_2 at p if and only if f has property c_3

at p. Notice that if f has property c_2 at the point p of f then f has property c_3 at p. Now suppose f has property c_3 at the point p of f. There exists a segment s which contains the abscissa of p such that f has property c_3 over s. For each number pair (u,v) let $T_1(u,v)$, $T_2(u,v)$, ... and $B_1(u,v)$, $B_2(u,v)$, ... denote two sequences which satisfy the three properties of the sequences in the definition of property c_3 over s. Suppose c is a number. For each positive integer n, let $t_n(c)$ denote the set to which x belongs if and only if there exists some positive integer $k \leq n$ such that x is in $[T_n(c + 1/k, c - 1/k)]^*$ but x is not in $\sum_{p=1}^{K} [B_n(c + 1/p, c - 1/p)]^*$, and let $b_n(c)$ denote the set to which x belongs if and only if there exists some positive integer $k \leq n$ such that x is in $[B_n(c + 1/k, c - 1/k)]^*$ but x is not in $\sum_{p=1}^{k} [T_n(c + 1/p, c - 1/p)]^*$. Denote by $T_n(c)$ the collection of components of $t_n(c)$ and by $B_n(c)$ the collection of components of $b_n(c)$. The sequences $T_1(c)$, $T_2(c)$, ... and $B_1(c)$, $B_2(c)$ ... satisfy the three properties of the sequences in the definition of property c_2 over s. Therefore if f has property c_3 at the point p of f then f has property c_2 at p.

It will now be established that if f has property c_2 at the point p of f, then f has property c_4 at p. Suppose that there is some point p of f such that f has property c_2 at p but f does not have property c_4 at p.

There exist two vertical lines $|h$ and $|k$ with p between them and two horizontal lines $\underline{\alpha}$ and $\underline{\beta}$ such that $\alpha > \beta$ and a subset T of the x-projection of points of f between $|h$ and $|k$ and above $\underline{\alpha}$ and a subset B of the x-projection of points of f between $|h$ and $|k$ and below $\underline{\beta}$ such that each point of T is a limit point of B and each point of B is a limit point of T and such that f has property c_2 over (h,k). Denote by α' and β' two numbers such that $\alpha > \alpha' > \beta' > \beta$. Denote by $T_1(\alpha')$, $T_2(\alpha')$ ... and $B_1(\alpha')$, $B_2(\alpha')$, ... and by $T_1(\beta')$, $T_2(\beta')$, ... and $B_1(\beta')$ $B_2(\beta')$, ... two pairs of sequences each satisfying the properties of the sequence pair in the definition of property c_2 over (h,k).

Suppose that u is a point of T and v is a point of B, then there exists a positive integer N such that if N' is a positive integer greater than N then u is in $[T_{N'}(\alpha')]^*$ and $[T_{N'}(\beta')]^*$ and v is in $[B_N{}'(\alpha')]^*$ and $[B_{N'}(\beta')]^*$. For each positive integer k, let t_k denote the set to which x belongs if and only if, for each positive integer $k' \geq k$, x is in $[T_{N+k'}(\beta')]^*$ and let b_k denote the set to which x belongs if and only if, for each positive integer $k' \geq k$, x is in $[B_{N+k'}(\alpha')]^*$.

If j is a positive integer, t_j and b_j are closed sets. No point of B is in t_j and no point of T is in b_j. Therefore, t_j does not cover T and b_j does not cover B. Hence, there exists an interval I, which contains a point of T in its interior but contains no points of $t_1 + b_1$.

I_1 contains a subset T_1 of T and a subset B_1 of B such that every point of T_1 is a limit point of B_1 and every point of B_1 is a limit point of T_1. Since $t_2 + b_2$ does not cover $T_1 + B_1$, there is a subinterval I_2 of I_1 which contains a point of T_1 in its interior but contains no points of $t_2 + b_2$. In this manner, a monotonic sequence I_1, I_2 ... can be constructed such that I_n contains no point of $t_n + b_n$. There is a point x which is common to I_1, I_2 ... . Since x is not in $t_1 + t_2$ ..., $f(x) \leq \beta'$ and since x is not in $b_1 + b_2 + \ldots$, $f(x) \geq \alpha'$. This constitutes a contradiction. Therefore, if f has property c_2 at the point p of f, then f has property c_4 at p.

Theorem I will now be used to show that if f has property c_4 at the point p of f, then f has property c_3 at p. Suppose f has property c_4 at the point p of f. There exist two vertical lines $|h$ and $|k$ with p between them such that if $\underline{\alpha}$ and $\underline{\beta}$ are horizontal lines, and T is a subset of [a,b] having the property that each point of f with abscissa in T is between $|h$ and $|k$ and above $\underline{\alpha}$ and $\underline{\beta}$ and B is a subset of [a,b] having the property that each point of f with abscissa in B is between $|h$ and $|k$ and below $\underline{\alpha}$ and $\underline{\beta}$, then there is a point of T which is not a limit point of B or there is a point of B which is not a limit point of T. The non-trivial case is the one in which there are points of f between $|h$ and $|k$ and above $\underline{\alpha}$ and $\underline{\beta}$ and points of f between $|h$ and $|k$

and below $\underline{\alpha}$ and $\underline{\beta}$. Denote by U the set of abscissas of all points of f between $|h$ and $|k$ and above $\underline{\alpha}$ and $\underline{\beta}$ and by V the set of abscissas of all points of f between $|h$ and $|k$ and below $\underline{\alpha}$ and $\underline{\beta}$.

Let G denote the collection of sets which satisfies the following three conditions:

(1) The collection G has property (f).

(2) The set U is the first member of G.

(3) The subcollection G' of G determines g means that the common part c of all the members of G' exists, there is a point of V in $\bar{c}$, there is a point of c in $\overline{V\cdot\bar{c}}$, and $g = c\cdot\overline{V\cdot\bar{c}}$.

Let H denote the collection of sets which satisfies the following three conditions:

(1) H has property (f).

(2) V is the first member of H.

(3) The subcollection H' of H determines h means that the common part d of all the members of H' exists, there is a point of U in $\bar{d}$, there is a point of d in $\overline{U\cdot\bar{d}}$ and $h = d\cdot\overline{U\cdot\bar{d}}$.

Notation. If K is a collection of pointsets and t is a point of a member of K or t is a pointset which is a subset of a member of K, then let K(t) denote the common part of all the members of K which contain t.

Let J_n denote the collection of intervals to which the interval I belongs if and only if I = [b+1,b+2] or the following three conditions are satisfied:

(1) The length of I is $1/2^n$.

(2) The center of I is a point x of U.

(3) If there is some point of V which is a limit point of G(x) and y is in $V \cdot \overline{G(x)}$ then $|x-y| > 4/2^n$.

Let K_n denote the collection of intervals to which the interval I belongs if and only if I = [a-2,a-1] or the following four conditions are satisfied:

(1) The length of I is $1/2^n$.

(2) The center of I is a point y of V.

(3) If there is a point of U which is a limit point of H(y) and x is in $U \cdot \overline{H(y)}$, then $|x-y| > 4/2^n$.

(4) If x is in $\overline{[J_n]^*}$ then $|x-y| \geq 3/2^n$.

Because of condition (4) in the definition of K_n, it follows that if n is a positive integer, $\overline{[J_n]^*}$ and $\overline{[K_n]^*}$ are mutually exclusive. Suppose u is in U and v is in V. It will now be established that there exists a positive integer n such that if n' is a positive integer greater than n, then u is in $[J_{n'}]^*$ and v is in $[K_{n'}]^*$. If no member of V is a limit point of G(u), then for each positive integer n, u is in J_n. Suppose that some member of V is a limit point of G(u). If u is a limit point of $V \cdot \overline{G(u)}$, then $G(u) \cdot \overline{V \cdot \overline{G(u)}}$ is a member of G which contains u and $G(u) = G(u) \cdot \overline{V \cdot \overline{G(u)}}$. Therefore, every point of $V \cdot \overline{G(u)}$ is a limit point of G(u) and every point of G(u) is a limit point of $V \cdot \overline{G(u)}$. This constitutes a contradiction. Therefore, u is not a limit point of

$V \cdot \overline{G(u)}$. Hence, there exists a positive integer n such that if n' is a positive integer greater than n, then no point of $V \cdot \overline{G(u)}$ is in $[u - 4/2^{n'}, u + 4/2^{n'}]$. Thus, u is the center of an interval of $J_{n'}$.

By the same reasoning, there exists a positive integer m such that if m' is a positive integer greater than m, then there is no point of U which is in $\overline{H(v)}$ and in $[v - 4/2^{m'}, v + 4/2^{m'}]$.

It will now be established that there exists a positive integer r such that if r' is a positive integer greater than r and x is in $\overline{[J_{r'}]^*}$, then $|x-v| \geq 3/2^{r'}$.

If v is not in $\overline{G(x)}$ for any x in U, then v is not in $\bar{U}$. Therefore, there exists a positive integer r such that if r' is a positive integer greater than r and y is in U, then $|y-v| > 4/2^{r'}$. Hence, if z is in $\overline{[J_{r'}]^*}$, then $|z-v| \geq 3/2^{r'}$.

If v is in $\overline{G(x)}$ for each x in U, and for some positive integer n, y denotes the center of an interval of J_n, then $|y-v| > 4/2^n$. Hence, if w is a point of $\overline{[J_n]^*}$, then $|w-v| \geq 3/2^n$.

If there are two numbers x and y in U such that v is in $\overline{G(x)}$ but v is not in $\overline{G(y)}$, then denote by M the set to which z belongs if and only if z is in U and v is not in $\overline{G(z)}$. If i is a positive integer and w denotes a member of an interval of J_i with center in [U-M], then $|w-v| > 3/2^i$. Either there is no set in G which is a proper subset of G(M) or there is some set in G which is a proper subset of G(M) and therefore, there is a first such set. In either case, there is a member x of M such that G(M) = G(x). Therefore, v is not a limit point of G(M) and consequently v is not a limit point of M. Hence, there exists a positive integer r such that if r' is a positive integer greater than r and Z denotes a member of an interval of $J_{r'}$ whose center is in M, then $|z - v| > 3/2^{r'}$. Therefore if Z denotes a member of an interval of $J_{r'}$ then $|z - v| > 3/2^{r'}$. Hence, there exists a positive integer s such that if s' is a positive integer greater than s then v is in $[K_{s'}]^*$.

For each positive integer n, let $T_n(\alpha,\beta)$ denote the collection of components of $\overline{[J_n]^*}$ and let $B_n(\alpha,\beta)$ denote the collection of components of $\overline{[K_n]^*}$. The sequences $T_1(\alpha,\beta)$, $T_2(\alpha,\beta)$, ... and $B_1(\alpha,\beta)$, $B_2(\alpha,\beta)$, ... satisfy the conditions imposed upon the sequences in the definition of property c_3 over (h, k). Therefore f has property c_3 at p. This completes the proof of Theorem II.

The fact that statements (1), (2), and (3) of Theorem III are equivalent and that statement (2) of Theorem III implies statement (4) of Theorem III can be established with arguments similar to those used in the proof of Theorem II. These arguments will be omitted from the paper. Theorem I will now be used to prove that if f has property r_4 at the point p of f then f has property r_3 at p. Suppose f has property r_4 at the point p of f. There exist two vertical lines $|h$ and $|k$ with p between them such that if $\underline{\alpha}$ and $\underline{\beta}$ are horizontal lines, and T is a subset of [A,B] having the property that each point of f with abscissa in T is between $|h$ and $|k$ and above $\underline{\alpha}$ and $\underline{\beta}$ and B is a subset of [a,b] having the property that each point of f with abscissa in B is between $|h$ and $|k$ and below $\underline{\alpha}$ and $\underline{\beta}$, then there is a point of T which is not a limit point of B from the right or there is a point of B which is not a limit point of T from the right. The non-trivial case is the one in which there are points of f between $|h$ and $|k$ and above $\underline{\alpha}$ and $\underline{\beta}$ and points of f between $|h$ and $|k$ and below $\underline{\alpha}$ and $\underline{\beta}$. Denote by U the set of abscissas of all points of f between $|h$ and $|k$ and above $\underline{\alpha}$ and $\underline{\beta}$ and by V the set of abscissas of all points of f between $|h$ and $|k$ and below $\underline{\alpha}$ and $\underline{\beta}$.

Notation. If M is a set, denote by $\overleftarrow{M}$ the set to which x belongs if and only if x is in M or x is a limit point of M from the right and denote by $\overrightarrow{M}$ the set to which x belongs if and only if x is in M or x is a limit point of M from the left.

Let G denote the collection of sets which satisfies the following three conditions:

(1) G has property (f).

(2) U is the first member of G.

(3) The subcollection G' of G determines g means that the common part of c of all the members of G' exists and

there is a point of V in $\overleftarrow{c}$ and there is a point of c in $\overleftarrow{V\cdot\overleftarrow{c}}$ and $g = c\cdot\overleftarrow{V\cdot\overleftarrow{c}}$.

Let H denote the collection of sets which satisfies the following three conditions:

(1) H has property (f).

(2) V is the first member of H.

(3) The subcollection H' of H determines h means that the common part d of all the members of H' exists and there is a point of U in $\overleftarrow{d}$ and there is a point of d in $\overleftarrow{U\cdot\overleftarrow{d}}$ and $h = d\cdot\overleftarrow{U\cdot\overleftarrow{d}}$.

For each positive integer n] let T_n denote the collection to which the set s belongs if and only if the following four conditions are satisfied:

(1) The set s is a sect closed on the left.

(2) The length of s is $1/n$.

(3) The left end of s is a point x of U.

(4) If y is in $V\cdot\overleftarrow{G(x)}$ and $x < y$ then $|x-y| \geq 4/n$.

Suppose x is a member of U. If there is a point of V in $\overleftarrow{G(x)}$ and x is in $V\cdot\overleftarrow{G(x)}$, then $G(x) = G(x)\cdot\overleftarrow{V\cdot\overleftarrow{G(x)}}$ and each point of $V\cdot\overleftarrow{G(x)}$ is a limit point of G(x) from the right and each point of G(x) is a limit point of $V\cdot\overleftarrow{G(x)}$ from the right. This constitutes a contradiction. Therefore x is not in $V\cdot\overleftarrow{G(x)}$. Hence, there exists a positive integer n such that if n' is a positive integer greater than n, x is the left end of a sect of $T_{n'}$.

Suppose for some positive integer n, J is a component of $[T_n]^*$. Let $\alpha(J)$ denote the collection to which s belongs if and only if the following five conditions are satisfied:

(1) The set s is a sect closed on the left.

(2) The length of s is $1/n$.

(3) The left end of s is a point y of $V \cdot J$.

(4) If x is in J, x is the left end point of a member of T_n, $|x-y| \leq 2/n$, $x < y$, and y is not a limit point of $G(x)$ from the left, then $|x-y| > 1/n$.

Denote by T'_n the collection to which c belongs if and only if c is a component of the set to which x belongs if and only if x is in a component K of T_n but x is not in $\overleftarrow{[\alpha(J)]^*}$. Notice that T'_n is finite and each member of T'_n is a segment or a sect closed on the left.

It will now be established that if i is a positive integer and x is a point of U which is the left end point of a sect of T_i, then the component of $(T_i)^*$ which contains x contains no point of U which is to the right of x and is in $\overleftarrow{G(x) \cdot V \cdot \overleftarrow{G(x)}}$. Suppose that the component J of $(T_i)^*$ which contains x contains a point w of U which is to the right of x and is in $\overleftarrow{G(x) \cdot V \cdot \overleftarrow{G(x)}}$. There is a positive integer k such that $k/i > |w-x|$. Since w is in $\overleftarrow{G(x) \cdot V \cdot \overleftarrow{G(x)}}$, there is a point y of $V \cdot \overleftarrow{G(x)}$ to the right of w such that $|y-w| < 1/i$. It follows from the definition of T_i that $|y-x| \geq 4/i$ and therefore, $|x-w| > 3/i$. There is a number sequence x_1, x_2, ... such that x_1 is the left end point of a sect of T_i.
$3/i > |x_1-x| \geq 1/i$, $x < x_1 < w$, $G(x)$ is a subset of $G(x_1)$,

and $|w-x_1| \geq 3/i$. Likewise, for each positive integer $n > 1$ there is a number $x_n > x_{n-1}$ such that x_n is the left end of a sect of T_i, $3/i > |x_n - x_{n-1}| \geq 1/i$, $x_{n-1} < x_n < w$, $G(x_{n-1})$ is a subset of $G(x_n)$, and $|w-x_n| \geq 3/i$. Therefore, $x_{k+1} > w$ but, for each p, $x_p < w$. This involves a contradiction.

It will now be established that if x is in U, there exists a positive integer p such that if q is a positive integer greater than p, then x is in T'_q. Suppose x is in U. We have seen that there exists a positive integer n_1 such that if m is a positive integer greater than n_1 then x is the left end point of a member of T'_m. For each positive integer $i > n_1$, let J_i denote the component of T_i which contains x. Suppose that there is some positive integer $i > n_1$ such that x is in $\alpha(J_i)$. Each point of J_i which is to the left of x and which is the left end point of a member of T_i has only one of the following two properties.

Property n. The point w has property n if and only if w is in U, x is not in $\overline{G(w)}$, and G(w) is a subset of G(x).

Property m. The point w has property m if and only if w is in U, x is in $\overline{G(w)}$, and G(w) is a subset of G(x).

Suppose that there is a point which is to the left of x, has property n, and is the left end point of a member of T_i. Let N denote the set of all such points. Either there is no element of G which is a proper subset of G(N) or there is a first such element. In either case there is a point y of N such that $G(y) = G(n)$. Therefore, x is not a limit point of $G(N)$. Hence, there exists a positive integer $n_2 > n_1$ such that if m is a positive integer greater than n_2 and w is in $G(N)$, then $|x-w| \geq 4/m$.

Suppose that there exists a positive integer $k > n_2$ such that x is in $\alpha(J_k)$. There is a sect s_y of $\alpha(J_k)$ whose left end point y is in $V \cdot J_k$ such that s_y contains x. There is a sect s_w in T_k whose left end point w is in J_k such that s_w contains y. The point w must have property m. Therefore, x is a limit point of $G(w)$. Hence there is a point of $G(w)$ in s_y. This constitutes a contradiction to property (4) of the definition of $\alpha(J_k)$. Therefore, for each positive integer $i > n_2$, x is not in $\alpha(J_i)$.

Suppose that for some positive integer $i > n_2$, x is in $[\alpha(J_i)]^*$. If x is a limit point of $G(x)$ from the right, then there is a point z of $G(x)$ such that $z > x$ and $|z-x| < 1/2i$. From property (4) of the definition of $\alpha(J_i)$, it follows that no point of $[\alpha(J_i)]^*$ lies between z and x. Therefore x is not a limit point of $G(x)$ from the right. Hence, there is a number ϵ such that no point of $G(x)$ is in $(x, x+\epsilon)$.

If y is some point of $J \cdot V$ such that $x < y$, y is the left end of a sect of $\alpha(J_i)$, and y is not in $G(x)$, then it follows from property (5) of the definition of $\alpha(J_i)$ that $|x-y| > 1/i$.

Denote by Q the set to which y belongs if and only if y is in $J \cdot V$, $x < y$, y is the left end of a sect of $\alpha(J_i)$, and y is in $\overrightarrow{G(x)}$. The point x must be a limit point of Q. Therefore there is a point z of Q such that z is in $(x, x+\epsilon)$. Hence, y is not in $\overrightarrow{G(x)}$. This constitutes a contradiction. It is therefore established that if m is a positive integer greater than n_2, then x is in T'_m.

It will now be established that if y is a point of V, then there exists a positive integer p such that if q is a positive integer greater than p then y is not in T'_q. Suppose that y is in V. We now introduce the following three definitions which depend upon y.

Property r. The point x has property r if and only if x is in U and y is not in $\overrightarrow{G(x)}$.

Property s. The point x has property s if and only if x is in U, y is in $\overrightarrow{G(x)}$, and y is not in $\overleftarrow{G(x)}$.

Property t. The point x has property t if and only if x is in U, y is in $\overrightarrow{G(x)}$, and y is in $\overleftarrow{G(x)}$.

Notice that if x is a point of U, then x has property r, s, or t. Suppose that there is some point of U with property r. Let R denote the set of all such points. Either there is no member of G which is a subset if G(R) or there is a first such set. In either case, there is a point z of R such that $G(Z) = G(R)$. Hence, y is not in $\overrightarrow{G(R)}$. Therefore, there exists a positive integer m_1 such that if n is a positive integer greater than m_1 and x is a point of G(R) to the left of y then $|x-y| \geq 4/n$.

If x is a point of U which has property s, $x < y$, and for some positive integer k, x is the left end point of a member of T_k, then it follows from part (4) of the definition of T_k that $|x-y| \geq 4/k$.

Suppose that there is some point of U which has property t. Let W denote the set of all such points. The point y is not in $V \cdot \overleftarrow{G(w)}$. Hence, there exists a positive integer $m_2 > m_1$ such that, for each positive integer $k > m_2$, there is no point of $G(w)$ in $[y, y + 1/k)$. If x is in J, x is the left end point of a member of T_k, $|x-y| \leq 2/k$, and $x < y$, then x has property t and $G(x)$ is a subset of $G(t)$ and no point of $G(x)$ is in $[y, y + 1/k)$. Hence, y is in $[\alpha(J_k)]^*$. Therefore, for each positive integer $m > m_2$, y is not in T'_m.

T'_1, T'_2, T'_3, ... is a sequence with the following two properties:

(1) If n is a positive integer, then T'_n is a finite collection each member of which is a segment or a sect closed on the left.

(2) If u is in U and v is in V, then there exists a positive integer J such that if K is a positive integer greater than J, then u is in T'_K and v is not in T'_K.

By a similar argument, it can be established that there exists a sequence B'_1, B'_2, B'_3, ... with the following two properties:

(1) If n is a positive integer, B'_n is a finite collection each member of which is a segment or a sect closed on the left.

(2) If u is in U and v is in V then there exists a positive integer J such that if K is a positive integer greater than J, then v is in B'_K and u is not in B'_K.

There is a positive integer n such that, for each positive integer $K > n$, there is a point of $(T'_K)^*$ which is not in $(B'_K)^*$ and a point of $(B'_K)^*$ which is not in $(T'_K)^*$. For each positive integer J, let T''_J denote the set to which x belongs if and only if x is in $(T'_{n+J})^*$ but x is not in $(B'_{n+J})^*$ and let B''_J denote the set to which x belongs if and only if x is in $(B'_{n+J})^*$ but x is not in $(T'_{n+J})^*$.

Suppose that i is a positive integer. If c is a component of T''_i and d is a component of B''_i, then there is a sect c' which is closed on the left and contains c but contains no point of B''_i and there is a sect d' which is closed on the left and contains d but contains no point of T''_i. Therefore, there exist two sequences $t_1, t_2, \ldots$ and $b_1, b_2, \ldots$ which satisfy the following three conditions:

(1) If n is a positive integer, each of t_n and b_n is a finite collection of closed on the left sects.

(2) If n is a positive integer, $(t_n)^*$ and $(b_n)^*$ are mutually exclusive.

(3) If u is in U and v is in V, then there exists a positive integer k such that, for each positive integer $j > k$, u is in $(t_j)^*$ and v is in $(b_j)^*$.

Therefore if f has property r_4 at the point p then f has property r_3 at p.

The vita has been removed from the digitized version of this document.